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Upper bounds to the correlation functions of the random-bond Ising model and *n*-vector model with competing interactions

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Abstract. We study a correlation function which is given by the canonical average of a product of one or more spin variables, for the random-bond Ising model, in which the exchange integrals are $+J$ $(J>0)$ and $-J$ with probabilities p and $1-p$, respectively. We show that an upper bound to the configurational average of the correlation function, calculated in the thermodynamic limit in the zero external field limit, is the product of the same correlation function for the corresponding ferromagnetic Ising model at the temperature under consideration and the same quantity at the temperature T_1 which is determined by the condition $T_1 = 2J/|k_B \ln[p/(1-p)]|$, where k_B is the Boltzmann constant. Applying this result to the spontaneous magnetisation, we see that the configurational average of the spontaneous magnetisation is zero for p satisfying $1-p_c \leqslant p \leqslant p_c$, where $p_c = 1/[1+\exp(-2J/k_B T_c)]$ and T_c is the critical temperature of the ferromagnetic Ising model. p_c equals 0.70711 for the square lattice and 0.60907 for the SC lattice. The results are given for the random-bond Ising model of an arbitrary spin S, and also for the diluted random-bond Ising models of an arbitrary spin S, with the pair interaction and with general interaction, and for the diluted and undiluted random-bond n-vector model.

1. Introduction

We are concerned with the random-bond Ising model in which the exchange integrals take $+J$ $(J>0)$ and $-J$ with probabilities p and $1-p$, respectively. The nature of the possible phases in this system has been of much interest lately. Apart from the discussions on existence or non-existence of the spin glass phase in this system, it is widely believed that there is no spontaneous magnetisation in a range of concentration. However, this has not been proved yet except for the trivial case of $p=\frac{1}{2}$ (Nishimori and Suzuki 1980). In the present paper, we prove that there is no spontaneous magnetisation for $1-p_c \leqslant p \leqslant p_c$ for the system on the loose packed lattice as well as on the close packed lattice where $p_c = 1/[1+\exp(-2J/k_B T_c)]$, k_B is the Boltzmann constant and T_c is the Curie temperature for the ferromagnetic Ising model on the respective lattices. This value of concentration is equal to that for a dissociation of frustrated plaquettes (Schuster 1979, Kolan and Palmer 1980).

In order to prove the existence of such concentration p_c, we prove an inequality for the thermodynamic limit of the spin correlation function in the zero external field limit. In recent works, the exact energy and an upper bound to the specific heat on 'Nishimori's line' satisfying $\exp(2J/k_B T) = p/(1-p)$ have been obtained (Nishimori

1980, Morita and Horiguchi 1980). Horiguchi (1981) showed that the same arguments apply to the diluted system with competing interactions. In that work, he gave an expression for the correlation function, and used it to show that when the external field is zero, the configurational average of the canonical average of a product of an even number of spin variables is positive on Nishimori's line for the undiluted system and on the corresponding line for the diluted system. The same expression shows that an upper bound to the configurational average of the canonical average of a product of a number of spin variables is given by a product of the canonical averages of the same quantity for the corresponding ferromagnetic Ising model under zero external field at temperatures T and $T_1 = 2J/|k_B \ln[p/(1-p)]|$, respectively. This upper bound is good for nothing when we consider the canonical average of a product of an odd number of spin variables. Thus the main purpose of the present paper is to show that a similar inequality is also valid for the quantity calculated in the thermodynamic limit in the zero external field limit, where the limit as $N \to \infty$ is taken first and then the limit as $h \to +0$ where N is the total number of sites and h is the external field. The inequality is given in the form applicable to the system of arbitrary spin S. Its extension to the diluted random-bond Ising model is also given.

We define the thermodynamic limit at $T = 0$ in the zero external field limit by such a procedure that the limit as $T \to 0$ is taken first keeping h/k_BT constant, then the limit as $N \to \infty$ and finally the limit as $h/k_BT \to +0$; an argument justifying this limiting procedure is given in appendix 1. In the thermodynamic limit defined in this way, the configurational average of the spin correlation function is shown to be bounded above by the thermodynamic limit of the spin correlation function for the ferromagnetic Ising model at temperature $T_1 = 2J/|k_B \ln[p/(1-p)]|$. Thus the existence of p_c at $T = 0$ is confirmed. This p_c is an exact lower bound for the critical concentration of the ferromagnetic bonds for the ferromagnetic ground state, and $1 - p_c$ is an exact upper bound for the critical concentration for the antiferromagnetic ground state.

In § 2, we discuss the consequence of the obtained inequalities. A proof of the inequalities is given in § 3. The theorems are generalised to the Ising model with a general interaction in § 4, and to the n-vector model in § 5. Concluding remarks are given in § 6. In the argument in § 3, we use theorems given in a previous paper by Horiguchi and Morita (1979, to be referred to as HM). In § 4, we need a theorem which is the extension of theorem 1 of HM for the random Ising model of general interaction. It is proved in appendix 2.

2. Inequalities and their consequences

We consider the Ising model of arbitrary spin S, in which the spin variables take values $-S, -S+1, \ldots, S$, as well as the ordinary Ising model in which the spin variables take values $+1$ and -1. The latter is equivalent to the former of spin $\frac{1}{2}$, and quantities for them are related to each other by trivial relations. In order to discuss these systems at the same time, we shall call the latter the Ising model of spin ± 1. In the following, S is a positive integer or a positive half-odd integer or ± 1. We consider the systems on a lattice which consists of N sites and of bonds connecting the nearest neighbour pairs of sites. We have a spin on each site of the lattice, and an interaction between two adjacent spins connected by a bond of the lattice.

A bond is denoted by the pair of sites (i, j) which are on both ends of it. In the following Σ_i and Π_i denote the summation and the multiplication, respectively, over all the sites of the lattice, and $\Sigma_{(i,j)}$ and $\Pi_{(i,j)}$ denote these over all the bonds of the lattice, if no restriction is stated.

We now consider a system of spin S, which is described by the Hamiltonian

$$H = -\sum_{(i,j)} J_{ij} s_i s_j - h \sum_i \mu_i s_i \tag{2.1}$$

where s_i is the spin variable for the site i. For the bond (i, j), J_{ij} is the exchange integral which is a quenched random variable and whose probability distribution is assumed to be given by

$$\tilde{P}(J_{ij}) = \begin{cases} p & J_{ij} = J > 0 \\ 1-p & J_{ij} = -J \end{cases} \tag{2.2}$$

independently of J_{kl} for the other bonds (k, l). h is the external field and μ_i is the magnetic moment of the spin on the site i. For a finite set A of sites in the system, the product of the spin variables for the sites in the set A is denoted by s_A:

$$s_A = \prod_{\substack{k \\ (k \in A)}} s_k. \tag{2.3}$$

In A, the same site k may occur more than once, and then the corresponding spin variable s_k must be multiplied repeatedly to give s_A. The canonical average of s_A, that is, the spin correlation function, is defined by

$$\langle s_A \rangle_{N,h}^{(\beta)} = \mathrm{Tr}\, s_A\, \mathrm{e}^{-\beta H} / \mathrm{Tr}\, \mathrm{e}^{-\beta H} \tag{2.4}$$

where $\beta = 1/k_B T$, T is the absolute temperature and k_B is the Boltzmann constant. In studying the correlation function $\langle s_A \rangle_{N,h}^{(\beta)}$ for a system of spin S, there occurs the same correlation function for the system of spin ± 1. In order to distinguish between them, we shall use the notation $\langle \sigma_A \rangle_{N,h}^{(\beta)}$ for the system of spin ± 1. σ_i, σ_A, etc will denote s_i, s_A, etc for this system.

The configurational average of a function $Q\{J_{ij}\}$ of the set $\{J_{ij}\}$ is denoted by the angular brackets with suffix c:

$$\langle Q\{J_{ij}\} \rangle_c = \sum_{\{J_{ij}\}} P\{J_{ij}\} Q\{J_{ij}\} \tag{2.5}$$

where

$$P\{J_{ij}\} = \prod_{(i,j)} \tilde{P}(J_{ij}) = \prod_{(i,j)} \frac{\exp(\beta_1 J_{ij})}{2\cosh(\beta_1 J)}. \tag{2.6}$$

β_1 and the associated temperature $T_1 = 1/k_B|\beta_1|$ are introduced by

$$\exp(2\beta_1 J) = p/(1-p). \tag{2.7}$$

For $h = 0$, we have

$$\langle \langle s_A \rangle_{N,h=0}^{(\beta)} \rangle_c = \sum_{\{J_{ij}\}} P\{J_{ij}\} \langle \sigma_A \rangle_{N,h=0}^{(\beta_1)} \langle s_A \rangle_{N,h=0}^{(\beta)} \tag{2.8}$$

in which $P\{J_{ij}\}$ is given by

$$P\{J_{ij}\} = \frac{1}{2^N} \sum_{\{\sigma_i\}} \prod_{(i,j)} \frac{\exp(\beta_1 J_{ij} \sigma_i \sigma_j)}{2\cosh(\beta_1 J)} \tag{2.9}$$

(Horiguchi 1981). Invoking theorem 1 and its extension to the Ising model of general spin S, given in Horiguchi and Morita (1979), we have for arbitrary N an upper bound to the configurational average of the zero-field spin correlation function:

$$|\langle\langle s_A\rangle_{N,h=0}^{(\beta)}\rangle_c| \leq \langle\sigma_A\rangle_{N,h=0}^{(|\beta_1|)(+J)}\langle s_A\rangle_{N,h=0}^{(\beta)(+J)} \tag{2.10}$$

where $\langle s_A\rangle_{N,h=0}^{(\beta)(+J)}$ and $\langle\sigma_A\rangle_{N,h=0}^{(|\beta_1|)(+J)}$ are the zero-field spin correlation function for the ferromagnetic Ising model of spin S at temperature T and that of spin ±1 at temperature T_1, respectively.

Inequalities similar to (2.10) are valid also for the thermodynamic limit of the spin correlation function in the zero external field limit. The detail of its proof is given in the next section and here we only use the results. First we have the following inequality from theorem 1 in the next section:

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} \leq \langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+J)}\langle s_A\rangle_{\mathrm{ThL}}^{(\beta)(+J)} \tag{2.11}$$

where $|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}}$, $\langle s_A\rangle_{\mathrm{ThL}}^{(\beta)(+J)}$ and $\langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+J)}$ are the thermodynamic limits of the absolute value of the spin correlation function in the zero external field limit for the random Ising model of general spin S, and for the ferromagnetic Ising model of general spin S and of spin ±1, respectively. They are defined by

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} = \lim_{h\to+0}\lim_{N\to\infty}|\langle\langle s_A\rangle_{N,h}^{(\beta)}\rangle_c| \tag{2.12}$$

$$\langle s_A\rangle_{\mathrm{ThL}}^{(\beta)(+J)} = \lim_{h\to+0}\lim_{N\to\infty}\langle s_A\rangle_{N,h}^{(\beta)(+J)} \tag{2.13}$$

$$\langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+J)} = \lim_{h\to+0}\lim_{N\to\infty}\langle\sigma_A\rangle_{N,h}^{(|\beta_1|)(+J)}. \tag{2.14}$$

The system size tends to infinity as $N\to\infty$, in Van Hove's way (Ruelle 1969). In particular, for spontaneous magnetisation, we have

$$\lim_{h\to+0}\lim_{N\to\infty}|\langle\langle s_i\rangle_{N,h}^{(\beta)}\rangle_c| \leq m_I(T_1)m_S(T) \tag{2.15}$$

where $m_I(T_1)$ and $m_S(T)$ are the spontaneous magnetisation of the ferromagnetic Ising model of spin ±1 at temperature T_1 and that of spin S at temperature T. For example, $m_I(T_1)$ is given by

$$m_I(T_1) = [1-16(1-p)^4p^4/(2p-1)^4]^{1/8} \tag{2.16}$$

$$m_I(T_1) = \{1-16(1-p)^6p^2/[(p^2+3(1-p)^2)(2p-1)^3]\}^{1/8} \tag{2.17}$$

for the square and the triangular lattice, respectively, when $p\geq\frac{1}{2}$, with the aid of the analytic expressions for spontaneous magnetisation (McCoy and Wu 1973, Potts 1952). For the cubic lattices, we are able to use the Padé approximant for the calculations of the upper bound $m_I(T_1)m_S(T)$. $m_I(T_1)$ is zero when $T_1\geq T_c$. Then we do not have spontaneous magnetisation in our random system at an arbitrary temperature $T>0$ if $1-p_c\leq p\leq p_c$. T_c is the Curie temperature for the ferromagnetic Ising model of spin ±1 and p_c is given by

$$p_c = 1/[1+\exp(-2J/k_BT_c)]. \tag{2.18}$$

$m_I(T_1)$ is shown in figure 1 for the square, triangular and SC lattices. The values of p_c for several lattices are given in table 1.

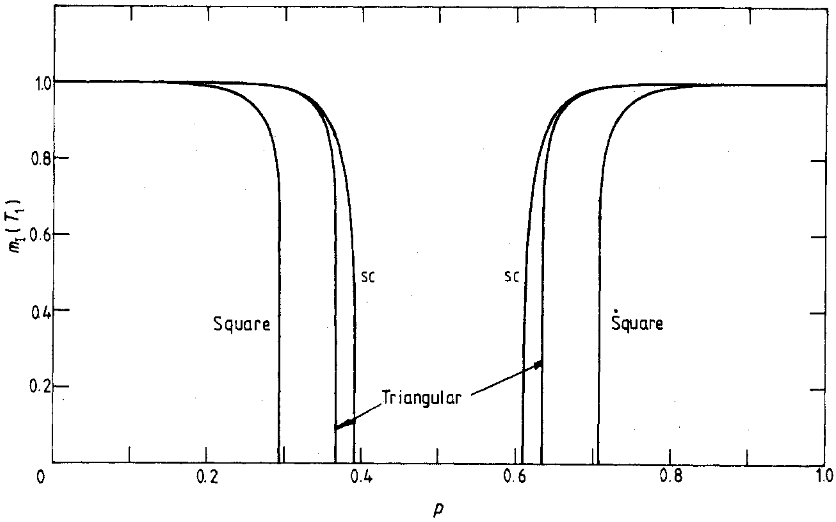

Figure 1. The upper bound to the spontaneous magnetisation divided by $|S|$ for the random-bond Ising model of arbitrary spin S and to the spontaneous magnetisation for the random-bond n-vector model, when the exchange interaction is $+J(J>0)$ and $-J$ with probabilities p and $1-p$, respectively.

Table 1. The lower bound to the critical concentration of the ferromagnetic bonds, p_c, for the ferromagnetic state, and the upper bound to the same quantity, $1-p_c$, for the antiferromagnetic state.

Lattice	p_c	$1-p_c$
Hexagonal	0.788 6751	0.211 3249
Square	0.707 1068	0.292 3932
Triangular	0.633 9747	0.366 0253
SC	0.609 07	0.390 93
BCC	0.578 06	0.421 94
FCC	0.550 87	0.449 13

When we use inequality (2.11), we have an upper bound to the susceptibility:

$$\chi_{S,c}^{(\beta)} \leq \min\left(S^2 \frac{\beta}{|\beta_1|} \chi_I^{(|\beta_1|)(+)}, \chi_S^{(\beta)(+)}\right) \tag{2.19}$$

where

$$\chi_{S,c}^{(\beta)} = \lim_{N_1\to\infty} \lim_{h\to+0} \lim_{N\to\infty} \beta \sum_{\substack{j\\(j\in\Gamma)}} (\langle\langle s_i s_j\rangle_{N,h}^{(\beta)}\rangle_c - \langle\langle s_i\rangle_{N,h}^{(\beta)}\langle s_j\rangle_{N,h}^{(\beta)}\rangle_c) \tag{2.20}$$

$$\chi_I^{(|\beta_1|)(+)} = \lim_{N_1\to\infty} \lim_{h\to+0} \lim_{N\to\infty} |\beta_1| \sum_{\substack{j\\(j\in\Gamma)}} \langle\sigma_i\sigma_j\rangle_{N,h}^{(|\beta_1|)(+J)} \tag{2.21}$$

$$\chi_S^{(\beta)(+)} = \lim_{N_1\to\infty} \lim_{h\to+0} \lim_{N\to\infty} \beta \sum_{\substack{j\\(j\in\Gamma)}} \langle s_i s_j\rangle_{N,h}^{(\beta)(+J)}. \tag{2.22}$$

Here i is an arbitrary site in the system and Γ is a set of sites including the site i. The total number of the sites N_1 in the set Γ tends to infinity in the sense of Van Hove (Ruelle 1969) at the last stage. The inequality (2.19) shows that the susceptibility $\chi_{S,c}^{(\beta)}$ never diverges when $1-p_c \leq p \leq p_c$, at any temperature $T>0$.

In order to investigate the properties at $T=0$, we define the thermodynamic limit at $T=0$ of the spin correlation function in the zero external field limit by

$$|\langle\langle s_A\rangle^{(\beta=\infty)}\rangle_c|_{\mathrm{ThL}} = \lim_{\beta h\to+0} \lim_{N\to\infty} \lim_{\beta J\to\infty} |\langle\langle s_A\rangle_{N,h}^{(\beta)}\rangle_c| \tag{2.23}$$

(see appendix 1). Here the spin correlation function is regarded as a function of N, βJ and βh. While keeping βh constant, the limit as $\beta J \to \infty$ is taken first and then the limit as $N\to\infty$. Finally we take the limit as $\beta h \to +0$. When we apply this limiting procedure to the ferromagnetic Ising model on the linear chain and the system consisting of free spins under uniform external field, we are able to obtain the right results at $T=0$ for these systems. In theorem 1 in the next section, we have

$$|\langle\langle s_A\rangle^{(\beta=\infty)}\rangle_c|_{\mathrm{ThL}} \leq \langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+J)} \tag{2.24}$$

for the spin correlation function and then

$$|\langle\langle s_i\rangle^{(\beta=\infty)}\rangle_c|_{\mathrm{ThL}} \leq m_{\mathrm{I}}(T_1). \tag{2.25}$$

Thus equation (2.25) shows that p_c given by (2.18) is the lower bound to the critical concentration of the ferromagnetic bonds for the ferromagnetic ground state. This value of p_c is equal to the one obtained by Schuster (1979) for a dissociation of frustrated plaquettes.

3. Lemmas, proposition and theorems

In this section, we present two theorems and their proofs with the aid of three lemmas and one proposition. Theorem 1 is for the random-bond Ising model, where the exchange integrals take values J and $-J$. Theorem 2 is its generalisation for the diluted systems. Consequences of the theorems are discussed in § 2.

Lemma 1. We consider a function Q of the spin variables, which is bounded by a finite value M: $|Q|\leq M$. If the external fields h_i on a finite number N_1 of sites i are changed by Δh_i and if Δh_i multiplied by the magnetic moment μ_i at the site are bounded by Δh for all the sites: $|\Delta h_i \mu_i| \leq \Delta h$, then the following inequality holds for the associated change $\Delta\langle Q\rangle$ of the canonical average of the quantity Q for a regular or a random Ising model

$$|\Delta\langle Q\rangle| \leq M[\exp(2N_1\beta\Delta h|S|)-1] \tag{3.1}$$

where $|S|$ is equal to S when the spins in the system are of spin S which is an integer or a half-odd integer, and to 1 when they are of spin ± 1.

Proof. Within the present proof, the canonical average of Q calculated for the systems in which the external fields in the sites i are h_i and $h_i' = h_i + \Delta h_i$ are denoted by $\langle Q\rangle_h$ and $\langle Q\rangle_{h'}$, respectively. Then

$$\langle Q\rangle_{h'} = \frac{\mathrm{Tr}\, Q\exp(-\beta H - \beta\Delta H)}{\mathrm{Tr}\exp(-\beta H - \beta\Delta H)} = \frac{\langle Q\exp(-\beta\Delta H)\rangle_h}{\langle\exp(-\beta\Delta H)\rangle_h}$$

where

$$H=-\sum_{(i,j)}J_{ij}s_is_j-\sum_i h_i\mu_is_i \qquad \Delta H=-\sum_i \Delta h_i\mu_is_i.$$

Now $\Delta\langle Q\rangle$ is expressed as

$$\Delta\langle Q\rangle=\langle Q\rangle_{h'}-\langle Q\rangle_h=\frac{\langle Q[\exp(-\beta\Delta H)-\langle\exp(-\beta\Delta H)\rangle_h]\rangle_h}{\langle\exp(-\beta\Delta H)\rangle_h}.$$

In the last member the first factor Q in the numerator is bounded by M. Since the total number of the sites on which $\Delta h_i\mu_i$ is non-zero is N_1 at most and $|\Delta h_i\mu_i|\leq\Delta h$ for all i, the denominator is greater than $\exp(-N_1\beta\Delta h|S|)$ and the absolute value of the second factor in the numerator is bounded by $\exp(N_1\beta\Delta h|S|)-\exp(-N_1\beta\Delta h|S|)$. As the result, we have the inequality (3.1).

Lemma 2. We consider a function Q of the spin variables, which is bounded by a finite value M: $|Q|\leq M$. We denote the canonical average of Q for a regular or a random Ising model consisting of N spins by $\langle Q\rangle^{(\beta)}_{N,(h,h_1)}$ when the external field is h_1 for a finite number N_1 of sites and h for all the other sites, and the temperature is $T=1/k_B\beta$. If the magnetic moments are bounded by a finite value for all the sites, the limiting value of $\langle Q\rangle^{(\beta)}_{N,(h,0)}$ when N tends to infinity and then h to zero is equal to the average in the thermodynamic limit, at non-zero temperature T:

$$\lim_{h\to+0}\lim_{N\to\infty}\langle Q\rangle^{(\beta)}_{N,(h,0)}=\lim_{h\to+0}\lim_{N\to\infty}\langle Q\rangle^{(\beta)}_{N,(h,h)}\equiv\langle Q\rangle^{(\beta)}_{\mathrm{ThL}}. \tag{3.2}$$

In the thermodynamic limit at $T=0$, we have

$$\lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J\to\infty}\langle Q\rangle^{(\beta)}_{N,(h,0)}=\lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J\to\infty}\langle Q\rangle^{(\beta)}_{N,(h,h)}\equiv\langle Q\rangle^{(\beta=\infty)}_{\mathrm{ThL}} \tag{3.3}$$

where the limit as $\beta J\to\infty$ is taken, keeping the product βh finite.

Proof. If we apply lemma 1 by putting $h_i=h$ for all the sites i, and $\Delta h_i=-h$ for the N_1 sites and $\Delta h_i=0$ for all the other sites, we have the inequality

$$|\langle Q\rangle^{(\beta)}_{N,(h,0)}-\langle Q\rangle^{(\beta)}_{N,(h,h)}|<M[\exp(2N_1\beta|h|\mu_M|S|)-1] \tag{3.4}$$

where μ_M is an upper bound of the absolute values of the magnetic moments in the system. We take the limit as $N\to\infty$ and $h\to+0$. Then we have the equality (3.2).

We take the limit as $\beta J\to\infty$, $N\to\infty$ and $\beta h\to+0$, in this order, in (3.4) to obtain (3.3).

Lemma 3. For the configurational average $\langle\langle Q\rangle^{(\beta)}_{N,(h,h_1)}\rangle_c$ of $\langle Q\rangle^{(\beta)}_{N,(h,h_1)}$ for random systems, as described in lemma 2, we have the following equality at non-zero temperatures T:

$$\lim_{h\to+0}\lim_{N\to\infty}\langle\langle Q\rangle^{(\beta)}_{N,(h,0)}\rangle_c=\lim_{h\to+0}\lim_{N\to\infty}\langle\langle Q\rangle^{(\beta)}_{N,(h,h)}\rangle_c\equiv\langle\langle Q\rangle^{(\beta)}\rangle_{c,\mathrm{ThL}}.$$

At $T=0$, we have

$$\lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J\to\infty}\langle\langle Q\rangle^{(\beta)}_{N,(h,0)}\rangle_c=\lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J\to\infty}\langle\langle Q\rangle^{(\beta)}_{N,(h,h)}\rangle_c\equiv\langle\langle Q\rangle^{(\beta=\infty)}\rangle_{c,\mathrm{ThL}}.$$

Proof. The difference

$$|\langle\langle Q\rangle^{(\beta)}_{N,(h,0)}\rangle_{c}-\langle\langle Q\rangle^{(\beta)}_{N,(h,h)}\rangle_{c}|$$

is overestimated by the configurational average of the left-hand side of (3.4) and hence by the right-hand side of (3.4). We take the limits in the obtained inequality to confirm this lemma.

Proposition. For the ferromagnetic Ising model of N spins under a uniform external field h, we denote the canonical average of a product of spin variables $Q \equiv s_A$ by $\langle Q\rangle_{N,h,B}$, where B denotes the boundary condition imposed. Then the average of Q in the thermodynamic limit, $\langle Q\rangle_{\mathrm{ThL}}$, is obtained by

$$\langle Q\rangle_{\mathrm{ThL}}=\lim_{h\to+0}\lim_{N\to\infty}\langle Q\rangle_{N,h,B} \tag{3.5}$$

irrespective of the boundary condition.

Note 1. In proving the following theorems, we use the equality (3.5) only for the boundary condition B_0 that the boundary spins are not coupled with an outer system, and for the boundary condition B_1 that the boundary spins are coupled with a plus spin ferromagnetically.

Note 2. This proposition was proved by Lebowitz and Martin-Löf (1972) in the case of the general Ising model of spin S with pair interactions. We assume this proposition in § 4, when interactions are more general.

In the statements in the following theorems, we use the following notations.

Notations. We pay attention to the product s_A of the spin variables for a finite set A of sites. We consider a random-bond Ising model of spin S. $|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}}$ denotes the absolute value of the configurational average of the canonical average of the quantity s_A of the system, in the thermodynamic limit in the zero external field limit at temperature $T=1/k_B\beta$; see (2.12). This quantity is denoted by $\langle\langle s_A\rangle^{(\beta)(+)}\rangle_{c,\mathrm{ThL}}$ when all the exchange integrals are replaced by their absolute values in the calculation of the canonical average. $\langle s_A\rangle^{(\beta)(+J)}_{\mathrm{ThL}}$ denotes the canonical average of the quantity s_A in the thermodynamic limit in the zero external field limit at the temperature $T=1/k_B\beta$, for the system of spin S, in which the exchange integrals for all the bonds are equal to J. Those quantities for the system of spin ±1 are expressed by the same expressions with s_A replaced by σ_A.

Theorem 1. For a random-bond Ising model of spin S, in which the exchange integrals are equal to $+J$ $(J>0)$ and $-J$ with probabilities p and $1-p$, respectively, we have the following inequality

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}}\leq\langle\sigma_A\rangle^{(|\beta_1|)(+J)}_{\mathrm{ThL}}\langle s_A\rangle^{(\beta)(+J)}_{\mathrm{ThL}} \tag{3.6}$$

where $T_1=1/k_B|\beta_1|$ is the temperature determined by the condition $\exp(2\beta_1 J)=p/(1-p)$. Here β is either finite or $\beta=\infty$.

Proof. We consider such a set Γ of a finite number N_1 of spins, which involves A as a subset. We denote the canonical average of s_A for the system of N spins in which the spins are of spin S, the external field is equal to zero for the spins belonging to Γ and to h for all the other spins, and the temperature is $T = 1/k_B\beta$, by $\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}$ where B_0 is the boundary condition that the boundary spins are not connected to an outer system. We can express the configurational average of this quantity as follows

$$\begin{aligned}\langle\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\rangle_c &= \sum_{\{J_{ij}\}}\left(\prod_{(i,j)}\frac{\exp(\beta_1 J_{ij}\sigma_i\sigma_j)}{2\cosh(\beta_1 J)}\right)\frac{\operatorname{Tr} s_A \exp(-\beta H_1)}{\operatorname{Tr}\exp(-\beta H_1)}\\ &= \sum_{\{J_{ij}\}}\left(\prod_{(i,j)}\frac{\exp(\beta_1 J_{ij}\sigma_i\sigma_j)}{2\cosh(\beta_1 J)}\right)\left(\prod_{\substack{k\\(k\in A)}}\sigma_k\right)\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\end{aligned} \tag{3.7}$$

where σ_i is either $+1$ or -1 for the site i belonging to the set Γ, and $\sigma_i = +1$ for all the other sites, and

$$-H_1 \equiv \sum_{(i,j)}\sigma_i\sigma_j J_{ij}s_i s_j + h\sum_{\substack{i\\(i\notin\Gamma)}}\mu_i s_i.$$

We multiply $\exp(\beta_1 h_1 \Sigma_{i(i\in\Gamma)}\sigma_i)$ on both sides of (3.7), take the summation with respect to $\{\sigma_i\}$ over all the possible 2^{N_1} sets of values of $\{\sigma_i\}$ and divide by $[2\cosh(\beta_1 h_1)]^{N_1}$. We then have

$$\langle\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\rangle_c = \sum_{\{J_{ij}\}}P\{J_{ij}\}\langle\sigma_A\rangle^{(\beta_1)}_{N_1,h_1,B_1}\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0} \tag{3.8}$$

where

$$P\{J_{ij}\} = \sum_{\{\sigma_i\}}\left(\prod_{(i,j)}\frac{\exp(\beta_1 J_{ij}\sigma_i\sigma_j)}{2\cosh(\beta_1 J)}\right)\left(\prod_{\substack{i\\(i\in\Gamma)}}\frac{\exp(\beta_1 h_1\sigma_i)}{2\cosh(\beta_1 h_1)}\right). \tag{3.9}$$

$\langle\sigma_A\rangle^{(\beta_1)}_{N_1,h_1,B_1}$ is the canonical average of σ_A in the system which is composed of N_1 spins of spin ±1 on the sites belonging to the set Γ, where the external field multiplied by the magnetic moment is h_1 for all the N_1 sites, the temperature is given by $T_1 = 1/k_B|\beta_1|$ and B_1 denotes the boundary condition that the spins which do not belong to Γ, and have a non-zero exchange integral with a spin belonging to Γ, are all plus one. According to whether β_1 is positive or negative, it represents the average for the system in which the set of the exchange intergrals is $\{J_{ij}\}$ or $\{-J_{ij}\}$.

By theorem 1 of HM and its extension to the system of an arbitrary spin S, the two averages in the summand on the right-hand side of (3.8) are overestimated by their respective values for the systems in which all the exchange integrals and the external fields multiplied by the magnetic moment at each site are replaced by their absolute values. If the latter are expressed by $\langle\sigma_A\rangle^{(|\beta_1|)(+J)}_{N_1,h_1,B_1}$ and $\langle s_A\rangle^{(\beta)(+J)}_{N,(h,0),B_0}$, we have

$$|\langle\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\rangle_c| \leq \langle\sigma_A\rangle^{(|\beta_1|)(+J)}_{N_1,h_1,B_1}\langle s_A\rangle^{(\beta)(+J)}_{N,(h,0),B_0}. \tag{3.10}$$

We take the limit as $N\to\infty$ and then as $h\to+0$ in this inequality. By using lemma 2 on the right-hand side and lemma 3 on the left-hand side, we obtain

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\text{ThL}} \leq \langle\sigma_A\rangle^{(|\beta_1|)(+J)}_{N_1,h_1,B_1}\langle s_A\rangle^{(\beta)(+J)}_{\text{ThL}} \tag{3.11}$$

for an arbitrary N_1 and h_1. We now take the limit as $N_1\to\infty$ and then as $h_1\to0$. By the proposition, we then obtain (3.6) for finite β.

In order to show (3.6) for $\beta=\infty$, we take the limit as $\beta J\to\infty$, $N\to\infty$ and $\beta h\to+0$, in this order, in (3.10) to obtain (3.11) for $\beta=\infty$, and then take the limit as $N_1\to\infty$ and then as $h_1\to0$. As the result, we have (3.6) for $\beta=\infty$.

Theorem 2. For the diluted random-bond Ising model of spin S, in which the exchange integrals take values $+J$ $(J>0)$, $-J$ and 0 with probabilities p, q and $r=1-p-q$, respectively, we have the following inequalities

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}}\leqslant\langle\langle\sigma_A\rangle^{(|\beta_1|)(+)}\rangle_{c,\mathrm{ThL}}\langle s_A\rangle^{(\beta)(+J)}_{\mathrm{ThL}} \tag{3.12}$$

and

$$\langle\langle\sigma_A\rangle^{(|\beta_1|)(+)}\rangle_{c,\mathrm{ThL}}\leqslant\langle\sigma_A\rangle^{(|\beta_1|)(+(1-r)J)}_{\mathrm{ThL}} \tag{3.13}$$

where the temperature $T_1=1/k_{\mathrm{B}}|\beta_1|$ is defined by the relation $\exp(2\beta_1 J)=p/q$. We also have the inequality

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}}\leqslant\langle\sigma_A\rangle^{(|\beta_1|)(+J)}_{\mathrm{ThL}}\langle\langle s_A\rangle^{(\beta)(+)}\rangle_{c,\mathrm{ThL}}. \tag{3.14}$$

In (3.12) and (3.14), β is either finite or $\beta=\infty$.

Proof. The present proof consists of the following alterations in the proof of theorem 1. Following Horiguchi (1981), a parameter β_2 is introduced by $\exp(2\beta_2 J^2)=pq/r^2$, and the factors in the first brackets on the middle and right-hand sides of (3.7) are replaced by

$$\prod_{(i,j)}\frac{\exp(\beta_2 J_{ij}^2+\beta_1 J_{ij}\sigma_i\sigma_j)}{1+2\exp(\beta_2 J^2)\cosh(\beta_1 J)}.$$

(3.9) is replaced by

$$P\{J_{ij}\}=\sum_{\{\sigma_i\}}\left(\prod_{(i,j)}\frac{\exp(\beta_2 J_{ij}^2+\beta_1 J_{ij}\sigma_i\sigma_j)}{1+2\exp(\beta_2 J^2)\cosh(\beta_1 J)}\right)\left(\prod_{\substack{i\\(i\in\Gamma)}}\frac{\exp(\beta_1 h_1\sigma_i)}{2\cosh(\beta_1 h_1)}\right).$$

In place of (3.10), we have

$$|\langle\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\rangle_c|\leqslant\sum_{\{J_{ij}\}}P\{J_{ij}\}\langle\sigma_A\rangle^{(|\beta_1|)(+)}_{N_1,h_1,B_1}\langle s_A\rangle^{(\beta)(+)}_{N,(h,0),B_0}. \tag{3.15}$$

Here $\langle\sigma_A\rangle^{(|\beta_1|)(+)}_{N_1,h_1,B_1}$ and $\langle s_A\rangle^{(\beta)(+)}_{N,(h,0),B_0}$ are the two averages on the right-hand side of (3.8) for the systems in which all the exchange integrals and the external fields multiplied by the magnetic moment at each site are replaced by their absolute values. Now we interchange the order of the summations over $\{J_{ij}\}$ and $\{\sigma_i\}$ and replace J_{ij} by $J_{ij}\sigma_i\sigma_j$, and then the summation over $\{\sigma_i\}$ is taken. In the result, we have

$$\prod_{(i,j)}\frac{\exp(\beta_2 J_{ij}^2+\beta_1 J_{ij})}{1+2\exp(\beta_2 J^2)\cosh(\beta_1 J)}$$

in place of $P\{J_{ij}\}$ in (3.15). We shall denote the value of $\langle s_A\rangle^{(\beta)(+)}_{N,(h,0),B_0}$, for the set $\{J_{ij}\}$ in which J_{ij} are all equal to J, by $\langle s_A\rangle^{(\beta)(+J)}_{N,(h,0),B_0}$, which is a common upper bound of $\langle s_A\rangle^{(\beta)(+)}_{N,(h,0),B_0}$. In confirming this fact, we use Griffiths' inequality (Griffiths 1977). We replace the last average of (3.15) by this upper bound and we obtain

$$\langle\langle s_A\rangle^{(\beta)}_{N,(h,0),B_0}\rangle_c\leqslant\langle\langle\sigma_A\rangle^{(|\beta_1|)(+)}_{N_1,h_1,B_1}\rangle_c\langle s_A\rangle^{(\beta)(+J)}_{N,(h,0),B_0}. \tag{3.16}$$

We take the thermodynamic limits as in the last part of the proof of theorem 1 to obtain (3.12), for finite β and for $\beta=\infty$. We obtain inequality (3.14), by exchanging the roles of the two averages in the summand of (3.15).

By theorem 2 of HM or Jędrzejewski's inequality (Jędrzejewski 1978) used in proving it, $\langle\sigma_A\rangle^{(|\beta_1|)(+)}_{N_1,h_1,B_1}$ is overestimated by the same quantity for the ferromagnetic Ising model, in which the exchange integral is equal to $\langle |J_{ij}|\rangle_{\mathrm{c}} = (1-r)J$; this fact is expressed as

$$\langle\langle\sigma_A\rangle^{(|\beta_1|)(+)}_{N_1,h_1,B_1}\rangle_{\mathrm{c}} \leqslant \langle\sigma_A\rangle^{(|\beta_1|)(+(1-r)J)}_{N_1,h_1,B_1}.$$

In the limit as $N_1 \to \infty$ and then as $h_1 \to +0$, we obtain (3.13).

4. Ising model of general random interactions

We consider the general Ising model of an arbitrary spin S, with the Hamiltonian

$$H = -\sum_{\rho} J_\rho s^\rho - h \sum_i \mu_i s_i \tag{4.1}$$

where ρ denotes a multiplicity function which associates a positive integer $\rho(i)$ to each site i, and $s^\rho \equiv \Pi_i s_i^{\rho(i)}$. The first summation on the right-hand side is taken over all the multiplicity functions. J_ρ is the exchange integral for the product s^ρ and its probability distribution is assumed to be given by

$$\tilde{P}_\rho(J_\rho) = \begin{cases} p_\rho & J_\rho = |J_\rho| \\ q_\rho & J_\rho = -|J_\rho| \\ r_\rho = 1 - p_\rho - q_\rho & J_\rho = 0 \end{cases} \tag{4.2}$$

independently of the other exchange integrals $J_{\rho'}$ for $\rho' \neq \rho$. We introduce parameters β_ρ and $\tilde{\beta}_\rho$ by

$$\exp(2\beta_\rho |J_\rho|) = p_\rho / q_\rho \tag{4.3}$$

$$\exp(2\tilde{\beta}_\rho |J_\rho|^2) = p_\rho q_\rho / r_\rho^2. \tag{4.4}$$

We have theorem 3 corresponding to theorem 2.

Theorem 3. Given the Hamiltonian (4.1) with (4.2), then

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_{\mathrm{c}}|_{\mathrm{ThL}} \leqslant \langle\langle\sigma_A\rangle^{\{|\beta_\rho|\}(+)}\rangle_{\mathrm{c,ThL}} \langle s_A\rangle^{(\beta)\{+|J_\rho|\}}_{\mathrm{ThL}} \tag{4.5}$$

$$\langle\langle\sigma_A\rangle^{\{|\beta_\rho|\}(+)}\rangle_{\mathrm{c,ThL}} \leqslant \langle\sigma_A\rangle^{\{|\beta_\rho|\}\{+(1-r_\rho)|J_\rho|\}}_{\mathrm{ThL}}. \tag{4.6}$$

We also have

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_{\mathrm{c}}|_{\mathrm{ThL}} \leqslant \langle\sigma_A\rangle^{\{|\beta_\rho|\}\{+|J_\rho|\}}_{\mathrm{ThL}} \langle\langle s_A\rangle^{(\beta)(+)}\rangle_{\mathrm{c,ThL}}. \tag{4.7}$$

In equations (4.5) and (4.7), β is either finite or $\beta = \infty$.

The following notes are given on the notations. σ_A is used when the average is for the system of spin ± 1. The thermodynamic limit is defined by taking the limit of the total number of sites infinity first and then the zero external field limit. The angular brackets with a superscript $(+)$ shows the canonical average with the Hamiltonian (4.1) where J_ρ takes two values $|J_\rho|$ and 0 with respective probabilities $1-r_\rho$ and r_ρ. The angular brackets with a superscript $\{+|J_\rho|\}$ and $\{+(1-r_\rho)|J_\rho|\}$ show the canonical averages for the ferromagnetic general Ising model of the exchange integrals $\{|J_\rho|\}$ and of the exchange integrals $\{(1-r_\rho)|J_\rho|\}$, respectively. When $\{|\beta_\rho|\}$ occurs in the superscript, we use the following Boltzmann factor in the calculation of the canonical average

$$\exp(-\beta H_1)/\mathrm{Tr}\,\exp(-\beta H_1)$$

where

$$-\beta H_1 = \sum_\rho |\beta_\rho| J_\rho s^\rho + \beta_1 \sum_i \mu_i s_i.$$

Here β_1 is an arbitrary positive number.

In order to prove this theorem, it suffices to mention the following points. The factors in the first brackets on the middle and right-hand sides of (3.7) and in the first brackets of (3.9) are replaced by

$$\prod_\rho \frac{\exp(\tilde{\beta}_\rho J_\rho^2 + \beta_\rho J_\rho \sigma^\rho)}{1 + 2\exp(\tilde{\beta}_\rho J_\rho^2)\cosh(\beta_\rho J_\rho)}. \tag{4.8}$$

$\Sigma_{\{J_{ij}\}}$ in (3.7), (3.8) and (3.15) are replaced by $\Sigma_{\{J_\rho\}}$, and $P\{J_{ij}\}$ in (3.9) and (3.15) by $P\{J_\rho\}$. The superscript $(|\beta_1|)$ of $\langle \sigma_A \rangle_{N_1,h_1,B_1}^{(|\beta_1|)(+)}$ and $\langle \sigma_A \rangle^{(|\beta_1|)(+(1-r)J)}$ should be replaced by the set $\{|\beta_\rho|\}$ of $|\beta_\rho|$, $(+J)$ of $\langle s_A \rangle_{N,(h,0),B_0}^{(\beta)(+J)}$ by the set $\{+|J_\rho|\}$ of $|J_\rho|$, and $(+(1-r)J)$ of $\langle \sigma_A \rangle_{N_1,h_1,B_1}^{(|\beta_1|)(+(1-r)J)}$ by $\{+(1-r_\rho)|J_\rho|\}$ of $(1-r_\rho)|J_\rho|$. In the present case, β_1 may be an arbitrary positive number and we take the limit as $N_1 \to \infty$ and then as $\beta_1 h_1 \to 0$ to obtain the thermodynamic limit $\langle\langle \sigma_A \rangle^{\{|\beta_\rho|\}(+)}\rangle_{c,\mathrm{ThL}}$ in equation (4.5). In the proof, we need lemma 3 which is based on lemmas 1 and 2, which are seen to be applicable also to the present case by an appropriate replacement of the Hamiltonian. The proposition given in § 3 is assumed here. We also need theorem 1 given by HM for the general Hamiltonian (4.1). This is given in appendix 2.

Now by setting $r_\rho = 0$, we have an extension of theorem 1 to the general Ising model.

Corollary. Given the Hamiltonian (4.1) with (4.2) setting $r_\rho = 0$, then

$$|\langle\langle s_A \rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} \leq \langle \sigma_A \rangle_{\mathrm{ThL}}^{\{|\beta_\rho|\}\{+|J_\rho|\}} \langle s_A \rangle_{\mathrm{ThL}}^{(\beta)\{+|J_\rho|\}} \tag{4.9}$$

where β_ρ is determined by $\exp(2\beta_\rho |J_\rho|) = p_\rho/(1-p_\rho)$. β is either finite or $\beta = \infty$.

Here we consider the system with the Hamiltonian

$$H = -\sum_{\substack{(i,j)\\(i,j:\mathrm{NN})}} J_{ij} s_i s_j - J_2 \sum_{\substack{(i,j)\\(i,j:\mathrm{NNN})}} s_i s_j - h \sum_i \mu_i s_i \tag{4.10}$$

where the first summation is taken over all nearest neighbour pairs of sites and the second one over all next nearest neighbour pairs of sites. $\tilde{P}(J_{ij})$ is assumed to be given by (2.2). In this case we have the following inequality from (4.9)

$$|\langle\langle s_A \rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} \leq \langle s_A \rangle_{\mathrm{ThL}}^{(\beta)\{+J,|J_2|\}}. \tag{4.11}$$

This is nothing but the upper bound obtained previously (Horiguchi and Morita 1979). Even for a small value of J_2, we cannot conclude from (4.11) that there is a critical concentration p_c below which the ferromagnetic state disappears. We rather expect that there is not such a critical concentration p_c in the system (4.10).

5. Random-bond n-vector model

In this section, we consider the n-vector model with competing interactions whose Hamiltonian is given by (Stanley 1974)

$$H = -\sum_{(i,j)} J_{ij} \mathbf{s}_i \mathbf{s}_j - h \sum_i \mu_i s_i^\alpha \tag{5.1}$$

where $\boldsymbol{s}_i$ is the n-dimensional classical spin of unit magnitude for the site i and s_i^α is its αth component:

$$\boldsymbol{s}_i = (s_i^1, s_i^2, \ldots, s_i^n) \qquad |\boldsymbol{s}_i| = 1. \tag{5.2}$$

J_{ij} for the bond (i, j) is the quenched random variable whose distribution is described by (2.2) independently of J_{kl} for the other bonds (k, l). The canonical average of a product of a number of spin variables, the spin correlation function, is defined by

$$\langle s_A\rangle_{N,h}^{(\beta)} = \mathrm{Tr}\, s_A\, \mathrm{e}^{-\beta H}/\mathrm{Tr}\, \mathrm{e}^{-\beta H} \tag{5.3}$$

where

$$s_A = \prod_{\substack{(i,\alpha_i)\\((i,\alpha_i)\in A)}} s_i^{\alpha_i}. \tag{5.4}$$

Here A represents a set of (i, α_i) where i denotes a spin in the system and α_i is its component.

For $h = 0$, we have the following equation for the configurational average of the spin correlation function

$$\langle\langle s_A\rangle_{N,h=0}^{(\beta)}\rangle_{\mathrm{c}} = \sum_{\{J_{ij}\}} P\{J_{ij}\}\langle\sigma_A\rangle_{N,h=0}^{(\beta_1)}\langle s_A\rangle_{N,h=0}^{(\beta)} \tag{5.5}$$

where $P\{J_{ij}\}$ is given by (2.9), and $\langle\sigma_A\rangle_{N,h_1}^{(\beta_1)}$ is the average $\langle s_A\rangle_{N,h_1}^{(\beta_1)}$ which would be obtained if the system is the Ising model of spin ± 1 and s_A is

$$s_A = \prod_{\substack{i\\((i,\alpha_i)\in A)}} s_i$$

or if the system is the n-vector model of $n = 1$ and s_A is

$$s_A = \prod_{\substack{i\\((i,\alpha_i)\in A)}} s_i^1.$$

Because $|s_i^\alpha| \leq 1$, we have

$$|\langle\langle s_A\rangle_{N,h=0}\rangle_{\mathrm{c}}| \leq \langle\sigma_A\rangle_{N,h=0}^{(|\beta_1|)(+J)} \tag{5.6}$$

instead of (2.10).

In order to obtain an upper bound to the thermodynamic limit of the spin correlation function in the zero external field limit defined by

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_{\mathrm{c}}|_{\mathrm{ThL}} = \lim_{h\to+0}\lim_{N\to\infty} |\langle\langle s_A\rangle_{N,h}^{(\beta)}\rangle_{\mathrm{c}}| \tag{5.7}$$

we notice that lemmas 1, 2 and 3 given in § 3 are also valid in the present system. We have

$$\langle\langle s_A\rangle_{N,(h,0),B_0}^{(\beta)}\rangle_{\mathrm{c}} = \sum_{\{J_{ij}\}} P\{J_{ij}\}\langle\sigma_A\rangle_{N_1,h_1,B_1}^{(\beta_1)}\langle s_A\rangle_{N,(h,0),B_0}^{(\beta)} \tag{5.8}$$

where $P\{J_{ij}\}$ is given by (3.9). Here the meanings of suffices of angular brackets are the same as those given in § 3. Because $|s_i^\alpha| \leq 1$, we have

$$|\langle\langle s_A\rangle_{N,(h,0),B_0}^{(\beta)}\rangle_{\mathrm{c}}| \leq \langle\sigma_A\rangle_{N_1,h_1,B_1}^{(|\beta_1|)(+J)}. \tag{5.9}$$

We take the limit as $N \to \infty$ and then as $h \to +0$ in this inequality. By using lemma 3 modified to the present system on the left-hand side of (5.9), we have

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_{\mathrm{c}}|_{\mathrm{ThL}} \leq \langle\sigma_A\rangle_{N_1,h_1,B_1}^{(|\beta_1|)} \tag{5.10}$$

for an arbitrary N_1 and h_1. We now take the limit as $N_1 \to \infty$ and then as $h_1 \to +0$, and we arrive at the following theorem.

Theorem 4. In the n-vector model with competing interactions whose distribution is given by (2.2), the thermodynamic limit of the spin correlation function in the zero external field limit defined by (5.7) is bounded above by the corresponding spin correlation function of the ferromagnetic n-vector model with $n = 1$ at the temperature T_1 given by (2.7):

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} \leqslant \langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+J)}. \tag{5.11}$$

Here β is either finite or $\beta = \infty$.

When $\beta = \infty$, the quantity on the left-hand side of (5.11) is defined by

$$|\langle\langle s_A\rangle^{(\beta=\infty)}\rangle_c|_{\mathrm{ThL}} = \lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J\to\infty}|\langle\langle s_A\rangle_{N,h}^{(\beta)}\rangle_c|. \tag{5.12}$$

The results obtained do not seem to be useful for the two-dimensional system with $n \geqslant 2$. However, for the three-dimensional system, these show that there is no spontaneous order when $1 - p_c \leqslant p \leqslant p_c$, where p_c is given in § 2 for the respective cases. An extension of theorem 2 to the n-vector model is also straightforward.

Theorem 5. In the n-vector model in which exchange integrals take values $+J (J > 0)$, $-J$ and 0 with probabilities p, q and $r = 1 - p - q$, respectively, we have the following inequality

$$|\langle\langle s_A\rangle^{(\beta)}\rangle_c|_{\mathrm{ThL}} \leqslant \langle\sigma_A\rangle_{\mathrm{ThL}}^{(|\beta_1|)(+(1-r)J)} \tag{5.11}$$

where the temperature $T_1 = 1/k_B|\beta_1|$ is defined by the relation $\exp(2\beta_1 J) = p/q$. Here β is either finite or $\beta = \infty$.

6. Concluding remarks

We have obtained an upper bound to the thermodynamic limit of the correlation function for the random-bond Ising model of general spins with competing interactions and for the random-bond n-vector model with competing interactions, for the undiluted as well as diluted cases. Here we consider the undiluted systems of ferro- and antiferromagnetic bonds with respective probabilities p and $1 - p$. For the Ising model of general spin S, our upper bound at temperature T consists of a product of the corresponding correlation function of the ferromagnetic Ising model of spin ± 1 at temperature T_1 and that of the spin S at the temperature T, where $T_1 = 2J/|k_B \ln[p/(1-p)]|$. For the n-vector model, our upper bound is equal to the corresponding correlation function of the ferromagnetic n-vector model of $n = 1$ at the temperature T_1. Since T_1 is a monotonically decreasing function of p for $\frac{1}{2} \leqslant p \leqslant 1$, we could obtain a lower bound p_c to the critical concentration at which the spontaneous magnetisation disappears, by setting $T_1 = T_c$ where T_c is the Curie temperature of the ferromagnetic Ising model of spin ± 1, because the upper bound to the spontaneous magnetisation disappears at p_c.

We wish to make a few remarks in order. First we notice that the obtained results are symmetric at $p = \frac{1}{2}$ in the p axis. When we consider the system on a loose packed lattice, the absolute values of the configurational average of the spin correlation functions are invariant with respect to a simultaneous change of $J_{ij} \to -J_{ij}$ and $p \to 1-p$, and $s_i \to -s_i$ and $\mu_i \to -\mu_i$ for all the sites on one of the sublattices. Then $1-p_c$ is an upper bound to the critical concentration of ferromagnetic bonds for the antiferromagnetic phase. For the close packed lattice, we expect that there is no long-range order for $0 \leqslant p \leqslant \frac{1}{2}$. However we have not succeeded in proving this yet.

In the present paper, we introduced a magnetic moment μ_i for each site i. By choosing the signs of μ_i, we see that the obtained upper bound to the spin correlation function is an upper bound when any staggered external field is applied. As a consequence, no long-range order of any antiferromagnetic phase is possible when our upper bound to the spontaneous magnetisation is zero.

Here we also mention the generalisations of our theorems to the configurational average of a product of correlation functions. In this case, similar inequalities are also obtainable for the respective systems. For example, we have

$$\left|\left\langle \prod_{l=1}^{k} \langle s_{A_l}\rangle^{(\beta)} \right\rangle_{\rm c}\right|_{\rm ThL} \leqslant \left\langle \prod_{l=1}^{k} \sigma_{A_l} \right\rangle_{\rm ThL}^{(|\beta_1|)(+J)} \prod_{l=1}^{k} \langle s_{A_l}\rangle_{\rm ThL}^{(\beta)(+J)} \tag{6.1}$$

for the Ising model of general spin S, where A_l are subsets of the set of N sites. For the n-vector model, we have

$$\left|\left\langle \prod_{l=1}^{k} \langle s_{A_l}\rangle^{(\beta)} \right\rangle_{\rm c}\right|_{\rm ThL} \leqslant \left\langle \prod_{l=1}^{k} \sigma_{A_l} \right\rangle_{\rm ThL}^{(|\beta_1|)(+J)}. \tag{6.2}$$

Here A_l are the sets of (i, α_i). When $p = \frac{1}{2}$, we have $T_1 = \infty$ and hence

$$\left|\left\langle \prod_{l=1}^{k} \langle s_{A_l}\rangle^{(\beta)} \right\rangle_{\rm c}\right|_{\rm ThL} = 0. \tag{6.3}$$

Applying this to the expression of susceptibility, we obtain

$$\chi = (1-q)/k_{\rm B}T \tag{6.4}$$

for the Ising model of general spin S and for the n-vector model, when $p = \frac{1}{2}$. Here q is Edwards–Anderson's order parameter for the respective systems (Edwards and Anderson 1975).

The same kind of remarks also apply to the systems of diluted random bonds.

Appendix 1.

The thermodynamic limit in the zero external field limit at zero temperature is defined by (2.23) in the text. For a finite system of N sites, we retain the contribution only of the ground state by putting $\exp(\beta \Delta E) \gg 1$, where ΔE is the difference between the total energies of the ground state and the first excited state. We take the limit as $\beta \to \infty$ before $N \to \infty$. In the calculation of the spontaneous magnetisation, we have to put a weight to one direction, lifting the degeneracy for the spatial rotation of the whole system. If the total magnetisation is M which is of the order of N, the Boltzmann factor in an external field h satisfying $\exp(M\beta h) \gg 1$ is used for this purpose. This inequality is satisfied for any infinitesimal βh if M tends to infinity. In order to make the effect of the external field only to direct the magnetisation to one direction, we take the limit as $\beta h \to +0$ after the limit as $N \to \infty$ is taken.

Appendix 2.

We extend theorem 1 given by Horiguchi and Morita (1979) to a more general class of Ising model. The result is used in § 4 in the text.

Theorem. Consider a finite set of lattice sites 1, 2, . . . , N. For each site i, we assign a spin variable s_i which may take on the values $-|S_i|, -|S_i|+1, \ldots, |S_i|$ where $|S_i|$ is a positive integer or a half-odd integer. Let $\rho(i)$ be a multiplicity function which may take zero or a positive integer for each site i, and define the product s^ρ of s_i by

$$s^\rho = \prod_i s_i^{\rho(i)}. \tag{A1}$$

For a system with the Hamiltonian

$$H = -\sum_\rho J_\rho s^\rho \tag{A2}$$

where the interactions J_ρ are quenched random variables and $-\infty < J_\rho < \infty$ and the sum is over different multiplicity functions, we have

$$-\langle s^\rho\rangle_{H^{(+)}} \leqslant \langle s^\rho\rangle_H \leqslant \langle s^\rho\rangle_{H^{(+)}} \tag{A3}$$

where $\langle s^\rho\rangle_H$ denotes the canonical average of s^ρ with the Hamiltonian H and $\langle s^\rho\rangle_{H^{(+)}}$ that with the Hamiltonian

$$H^{(+)} = -\sum_\rho |J_\rho| s^\rho. \tag{A4}$$

Proof. We define an auxiliary Hamiltonian H' by

$$H' = -\sum_{\rho\in\Omega} |J_\rho| s^\rho s' - \sum_{\rho\notin\Omega} J_\rho s^\rho \tag{A5}$$

where Ω is the set of multiplicity functions ρ for which J_ρ is negative:

$$\Omega = \{\rho | J_\rho < 0\}. \tag{A6}$$

s' is a ghost spin variable of spin ± 1 on a ghost site. For the system of the Hamiltonian H', we have the second GKS inequality (Griffiths 1969) which reads in special cases as follows

$$\langle s^\rho s'\rangle_{H'} \geqslant \langle s^\rho\rangle_{H'} \langle s'\rangle_{H'} \tag{A7}$$

$$\langle s^\rho\rangle_{H'} \geqslant \langle s^\rho s'\rangle_{H'} \langle s'\rangle_{H'} \tag{A8}$$

where $\langle Q\rangle_{H'}$ is the canonical average of a product Q of the spin variables for the system in which s' is also a spin variable and the Hamiltonian is H'. The first part of the inequality (A3) is obtained from (A8) and the second part from (A7).

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